Phase measurement strategies in the presence of dephasing: The Ramsey experiment

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We discuss the Ramsey experiment in terms of the phase estimation algorithm and show different strategies for achieving high accuracy in presence of dephasing.

I. INTRODUCTION

- explain connection between the Ramsey experiment and PEA - so that we can use PEA terminology in the rest of the paper (and possibly keeping the impression that the Ramsey experiment and the PEA are interchangable terms in a given context)
- two real world applications - Yale qubit calibration and something with a cavity, Göran knows; optical interferometry might be a third application?
- atomic clock approach
- the issues of number of measurement/reps vs. real time costs - this is two fold: first, some people still feels being tricked when it is said that IPEA achieves n bits of precission after $O(n)$ rounds compared to $O(2^{2n})$ rounds with the Naive PEA. So we should be clear about that. Second, in our paper real time plays a big role. In fact, our aim is to optimize towards minimal time costs, not to a minimal number of rounds/measurements as such.

II. MODEL AND APPROACHES

- This section should start with the description of the model, but so far the model is part of Naive PEA subsection - Eq.(1).
- It is probably better to avoid drawing the gate model - the picture would be too simple and additionaly it will unnecessary emphasize the problem of implementing the Hadamard gates. I'd rather postpone this problem to the Appendix. So we use only the Hamiltonian $H = gZ/2$ in the description. Does such a description serve as a simplified model of the real word problems mentioned in the introduction at least somehow?

A. Naive PEA

This section explains the Naive Phase Estimation Algorithm. We have a qubit whose evolution is described as

$$U = e^{(-igtZ/2)}, \tag{1}$$

that is the qubit rotates around the Z-axis with angular speed g during time t. Measuring the qubit which rotates in the equatorial plane in the dual basis (X observable) gives the probability

$$P_{|+\rangle} = \cos^2(gt/2) \tag{2}$$

to observe the $|+\rangle$ state, and

$$P_{|-\rangle} = \sin^2(gt/2) \tag{3}$$

to observe the $|-\rangle$ state. The expectation value is then given by:

$$\langle X \rangle = \cos^2(gt/2) - \sin^2(gt/2) = \cos(gt)\,. \tag{4}$$

Tricky point: having $P_{|+\rangle} \pm 1/\sqrt{M} \;\Rightarrow\; \langle X \rangle \pm 2/\sqrt{M}$.

In presence of dephasing the formulas are modified to

$$\begin{aligned} P^d_{|+\rangle} &= \frac{1}{2}(1 + e^{-\gamma t}\cos(gt)), & (5)\\ P^d_{|-\rangle} &= \frac{1}{2}(1 - e^{-\gamma t}\cos(gt)), & (6)\\ \langle X^d \rangle &= e^{-\gamma t}\cos(gt)\,. & (7) \end{aligned}$$

Knowing a raw estimate $g_{est} \pm \Delta_g$ of angular speed g and dephasing rate $\gamma \pm \Delta_\gamma$, our task is to determine g with high precission. After M rounds of setup preparation, evolution and subsequent measurement we have an estimate $\langle X^d \rangle \pm 2/\sqrt{M}$ using the classical statistics. Now, the question is how the uncertainity in the expectation value and dephasing rate project into

$$\pi\phi = \mathrm{acos}\left(e^{(\gamma \pm \Delta_\gamma)t}\left(\langle X^d \rangle \pm 2/\sqrt{M}\right)\right)\,, \tag{8}$$

where $0 \leq \phi \leq 1$ represents all the information we get out of the experiment. *Note: working with $P^d_{|+\rangle}$ would give us exactly the same formula - replacing $\langle X^d \rangle$ with* $(2P^d_{|+\rangle} - 1)$. First order approximation gives us (plus-minus signs should be cleared up)

$$\begin{aligned} \pi\phi \approx\; & \mathrm{acos}\left(\langle X \rangle\right) & (9)\\ & \pm F \langle X^d \rangle t\Delta_\gamma \pm \frac{2F}{\sqrt{M}} \pm \frac{2(F^3 \langle X^d \rangle^2 + F)t\Delta_\gamma}{\sqrt{M}} \end{aligned}$$

where

$$F = \frac{e^{\gamma t}}{\sqrt{1 - \langle X \rangle^2}} > 1\,. \tag{10}$$

Under the conditions that $|\langle X\rangle| \ll 1$ and $t\Delta_\gamma \ll 1$ we have

$$F \gg \left(F^3 \left\langle X^d\right\rangle^2 + F\right) t\Delta_\gamma \,,$$

and thus Eq.(9) can be simplified to

$$\pi\phi \approx \mathrm{acos}\left(\langle X\rangle\right) \;\pm\; F\left\langle X^d\right\rangle t\Delta_\gamma \;\pm\; \frac{2F}{\sqrt{M}} \,. \tag{11}$$

The goal of knowing $\pi\phi$ with precission $\pm\frac{1}{N}$ is then possible only if

$$F\left|\left\langle X^d\right\rangle\right| t\Delta_\gamma + \frac{2F}{\sqrt{M}} \le \frac{1}{N} \,. \tag{12}$$

Note: Feels like capitals M, N are less worry-inducing at this stage compared to $2^m, 2^n$, for some people.

While the term $2F/\sqrt{M}$ can be made vanishingly small, in principle, the term with Δ_γ represents an essential obstacle. It must be that

$$F\left|\left\langle X^d\right\rangle\right| t\Delta_\gamma < \frac{1}{N} \,. \tag{13}$$

Note: otherwise we are simply measuring very accurately the wrong thing.

Now, assuming that the condition Eq.(13) is fulfilled, we get a lower bound on M

$$M \ge \left(\frac{2FN}{1 - NF\left|\langle X^d\rangle\right| t\Delta_\gamma}\right)^2 \,, \tag{14}$$

which can be kept small with respect to the troublesome Δ_γ only if Eq.(13) is strengthened to

$$F\left|\left\langle X^d\right\rangle\right| t\Delta_\gamma \ll \frac{1}{N} \,. \tag{15}$$

Further steps towards keeping M small involve minimization of the factor F. It is evident that the factor F grows enormously for all but $\gamma t < 1$ and $\langle X\rangle$ close to 0. From Eq.(4) and Eq.(10) we get

$$F = \frac{e^{\gamma t}}{\sin(gt)} \,. \tag{16}$$

This implies that t should be the shortest possible time satisfying

$$gt \pmod{\pi} = \frac{\pi}{2} \,. \tag{17}$$

In another words, $\pi/2$ is the optimal working point since it is one of the points having highest slope in Eq.(7) and therefore the crucial ratio $g/g_{est} < 2$ manifests its magnitude in the largest possible effect.

Let us write $g = g_{est} \pm \delta g$ and expand F as

$$F = \frac{e^{\gamma t}}{\cos(\pi\delta g/(2g_{est}))} = \frac{e^{\gamma t}}{\cos\left(\pi/2^\ell\right)} \,, \tag{18}$$

where $\ell > 1$ denotes the distance from the optimal working point. Then given the initial uncertainity $\Delta_g \ge \delta g$, we know that ℓ is at least $-\log_2(\Delta_g/(2g_{est}))$ and we immediately see that $\ell \ll 2$ yields a serious hardship since it makes F very large. On the other hand with $\ell \approx 2$ or 3 we are on the safe side and for $\ell \ge 4$ we can write $F = e^{\gamma t}$.

Final result. Given that the required precission $1/N$ satisfies Eq.(13), time is set to $t = \pi/(2g_{est})$, $\langle X\rangle$ is the outcome from the measurement, then following Eq.(11) we get:

$$\mathrm{acos}(\langle X\rangle) - \frac{1}{N} \lessapprox \;\pi\phi\; \lessapprox \mathrm{acos}(\langle X\rangle) + \frac{1}{N} \,, \tag{19}$$

$$2\phi - \frac{2}{\pi N} \lessapprox \;\tfrac{g}{g_{est}}\; \lessapprox 2\phi + \frac{2}{\pi N} \,, \tag{20}$$

$$2\phi - \frac{1}{N} < \;\tfrac{g}{g_{est}}\; < 2\phi + \frac{1}{N} \,. \tag{21}$$

Note: Since $\pi/2$ has been chosen as the optimal working point and $\delta g < g_{est}$, ϕ is now restricted into the sharp range $0 < \phi < 1$ and expected to be close to $1/2$.

The last inequality gives as a new estimate for g:

$$g_{est}^{new} = \left(2\phi \pm \frac{1}{N}\right) g_{est}^{old} \,, \tag{22}$$

and we conclude that the former ratio Δ_g/g_{est} is lifted to $\Delta_g^{new}/g_{est}^{new} \approx 1/N$.

The obstacle and penalty arising from Δ_γ in conjuction with a very large N, see Eq.(13) and Eq.(15), respectively, can be solved by pre-runs of NPEA with successively increasing required precission up to $1/N$. In this manner we can ensure that $F\left|\left\langle X^d\right\rangle\right|$ is always small enough in order to compensate for a large product $\Delta_\gamma N$. We talk about an adaptation towards the point of highest slope. Adaptation can also be used to make $\ell \ge 4$ which pays off for large N.

B. Iterative PEA

- IPEA is more naturally explained using $P_{|+\rangle}/P_{|-\rangle}$ instead of $\langle X\rangle$ as opposed to the Naive PEA.

The standard quantum Fourier transform based phase estimation algorithm (PEA) [1] represents a qualitatively different approach for phase estimation which is based on accumulation of the phase in well defined quantities. On the top of that, iterative variants reduce the number of needed qubits. The main idea is to utilize a single working qubit, execute a specific procedure over the relevant parts of the phase $\pi\phi$ and obtain the phase in a bit-by-bit fashion with high probability.

The basic iterative scheme [2, 3] can be derived either through replacing the quantum Fourier transform with its semi-classical counterpart [4] in the standard PEA or via introducing a feedback mechanism into the Kitaev algorithm [5]. The feedback can be additionally optimized with respect to the desired functionality using various strategies such as majority voting [6] and bayesian algorithm [7].

Assuming a general knowledge of (iterative) phase estimation algorithm, the k-th step of IPEA within the framework of our model can be described as follows. Let us write

$$\frac{g}{g_{est}} = \frac{g_{est} \pm \delta g}{g_{est}} = \phi_0.\phi_1\phi_2\phi_3\ldots\phi_m + \Delta_\phi/2^m\,, \quad (23)$$

where $0 \le \Delta_\phi < 1$ is a general reminder, and let the time be set to

$$t = \frac{\pi 2^k - \omega_k}{g_{est}}\,. \quad (24)$$

The term $\pi 2^k$ is responsible for the phase accumulation and $\omega_k = 2\pi(0.0\phi_{k+1}\ldots\phi_m)$ represents a feedback. Then

$$P_{|+\rangle} = \cos^2\Big(\pi 2^{k-1}\frac{g}{g_{est}} - \pi(0.0\phi_k\ldots\phi_m)\frac{g}{g_{est}}\Big)\,. \quad (25)$$

It is now convenient to upperbound $\pi(0.0\phi_{k+1}\ldots\phi_m)$ by $\pi/2$. Thus Eq.(25) can be rewritten in terms of $\phi_k, 0 \le k \le m$, as

$$\begin{aligned} P_{|+\rangle} &\ge \cos^2\Big(\pi\Big(0.\phi_k + \frac{\Delta_\phi}{2^{m-k+1}} \mp \frac{\delta g}{2g_{est}}\Big)\Big) & (26)\\ &\ge \cos^2\Big(\pi\Big(0.\phi_k \pm 2^{-\ell_k}\Big)\Big) & (27)\end{aligned}$$

for a largest possible integer ℓ_k satisfying

$$1 + \lfloor -\log_2 \Delta_\phi \rfloor \le \ell_k \le m - k + 1 + \lfloor -\log_2 \Delta_\phi \rfloor,$$

and $\ell_k \le \lfloor -\log_2(\delta g/(2g_{est}))\rfloor$. The range of ℓ_k is justified by observing that as long as

$$\frac{\delta g}{2g_{est}} \le 2\cdot\frac{\Delta_\phi}{2^{m-k+1}} \quad (28)$$

the term $\delta g/(2g_{est})$ actually supports a proper up/down rounding and thus helps to push $P_{|+\rangle}$ and $P_{|-\rangle}$ further apart. In other cases $\delta g/(2g_{est})$ represents a serious barrier in performing ever more accurate feedback. Let us call $2^{-\ell_k}$ the effective reminder and let $2^{-\ell} = \min_k\{2^{-\ell_k}\}$ denote the minimum over the set of effective reminders. We require $\ell \ge 3$ (i.e. $\frac{\Delta_g}{g_{est}} \le \frac{1}{4}$, since $\delta g \le \Delta_g$) to ensure a non-trivial feedback capabilities. If needed, such a minimal separation between g_{est} and Δ_g is easily obtained using the Naive PEA.

Note: The above introduction of the effective reminder ℓ_k allows as to have a single reference point irrespectively of the dominant term in Eq.(26). A loose reader might not notice properly the dependency of ℓ_k on Δ_ϕ and then be confused by Eq.(29).

Now, if we identify the output state $|+\rangle$ with the result $x_k = 0$ and $|-\rangle$ with the result $x_k = 1$, we get the conditional bitwise probability (conditioned on ω_k) that $x_k = \phi_k$ as

$$P_k(\Delta_\phi) \ge \cos^2\big(\pi/2^{\ell_k}\big)\,. \quad (29)$$

Iterating the above described k-th step from $k = m$, with $\omega_m = 0$, down to $k = 0$ the algorithm returns the final assembled output $\tilde{\phi} = x_0.x_1x_2\ldots x_n$ such that

$$\left|\frac{g}{g_{est}} - \tilde{\phi}\right| < \frac{1}{2^n}\,, \quad (30)$$

where $1 \le n \le m$ and $n \le 2^\ell$, with probability

$$P_{\tilde{\phi}} = \prod_{k=1}^{n} P_k(\Delta_\phi) + \prod_{k=1}^{n} P_k(1-\Delta_\phi) \ \ge\ 8/\pi^2 \approx 0.81\,. \quad (31)$$

The left and right operand in the sum stand for the probability to observe the n-bit round down and round up version of g/g_{est}, respectively. The sum achieves its minimum for $\Delta_\phi = 1/2$, that is when neither the rounded up nor the rounded down outcome takes any precedence.

The sufficiency of ℓ being logaritmical in n in order to closely match the overall success probability of the textbook PEA has been proven by D. Cheung [8]. Obviously it makes no sense to start the IPEA with $m > 2^l$.

In presence of dephasing, Eq.(27) and Eq.(29) are modified to

$$\begin{aligned} P^d_{|+\rangle}(\Delta_\phi) &\ge \frac{1}{2}\Big(1 + e^{-\gamma t}\cos\big(2\pi\big(0.\phi_k \pm 2^{-\ell_k}\big)\big)\Big)\,, & (32)\\ P^d_k(\Delta_\phi) &\ge \frac{1}{2}\Big(1 + e^{-\gamma t}\cos\big(2\pi/2^{\ell_k}\big)\Big)\,. & (33)\end{aligned}$$

That is, the probability we are interested in is exponentially suppressed.

One of the fastest methods how to amplify it to its original strength is to use the Chernoff bound. The probability of the result x_k to be correctly determined after M_k independent trials is

$$P'_k(\Delta_\phi) = \sum_{s=0}^{\lfloor\frac{M_k-1}{2}\rfloor} \binom{M_k}{s} \big(P^d_k(\Delta_\phi)\big)^{N_k-s}\big(1 - P^d_k(\Delta_\phi)\big)^s\,, \quad (34)$$

given that x_k is chosen using majority voting from the outcomes. Thanks to the symmetry between $P^d_k(\Delta_\phi)$ and $P^d_k(1-\Delta_\phi)$ it is very convenient to restrict Δ_ϕ into the range $0 \le \Delta_\phi \le 1/2$ and avoid keeping track of $P^d_k(\Delta_\phi)$ and $P^d_k(1-\Delta_\phi)$ separately.

Calculating the smallest possible integer M_k such that

$$P'_k(\Delta_\phi) \ge P_k(\Delta_\phi) \quad (35)$$

is difficult, but it is easy to find a lower bound on M_k. A lower bound that is sufficiently good for all practical purposes is

$$\begin{aligned} M_k(\Delta_\phi) &\ge \frac{e^{2\gamma t}}{\sqrt{P_k(\Delta_\phi)\big(1 - P_k(\Delta_\phi)\big)}} & (36)\\ &= \frac{2e^{2\gamma t}}{\sin\big(2\pi/2^{\ell_k}\big)}\,. & (37)\end{aligned}$$

Note: I found this bound by suggest, check & improve, but its form foreshadows that one might get the same one from the variance of binomial distribution.

In practice, since ℓ_k depends on Δ_ϕ and the ratio $\delta g/g_{est}$, both unknown, we cannot guarantee that $P'_k(\Delta_\phi)$ is higher or equal to $P_k(\Delta_\phi)$ in every situation. Assuming the worst case scenario where $\Delta_\phi = 1/2$ and $\delta g = \Delta_g$, we set ℓ_k to the largest possible integer such that

$$2 \le \ell_k \le m - k + 2 \ \text{ and } \ \ell_k \le \lfloor -\log_2\left(\Delta_g/(2g_{est})\right)\rfloor ,$$

and hereby we ensure that the probability $P'_k(\Delta_\phi)$ is always at least as high as $P_k(\Delta_\phi = 1/2)$.

The overall success probability $P'_{\tilde{\phi}}$ then amounts to approximately 0.81. Accordingly, setting ℓ_k as if $\Delta_\phi = 1/4$ brings $P'_{\tilde{\phi}}$ into the range $[0.81, 0.9]$, and so on and so forth. However, bringing the success probability $P'_{\tilde{\phi}}$ back to the full range $[0.81, 1]$ in this manner is problematic as M_k grows with small Δ_ϕ rather fast and ℓ_k is constrained by the ratio $\Delta_g/(2g_{est})$ anyway.

Opportune method for amplifying the success probability $P'_{\tilde{\phi}}$ up to $1-\varepsilon$ for $\varepsilon > 0$ is either to repeat the whole experiment $O(\log(1/\varepsilon))$ times or keeping only $n' = n - O(\log(1/\varepsilon))$ most significant bits from $\tilde{\phi}$.

Since M_n represents dominating costs in estimating $\tilde{\phi} = x_0.x_1x_2\ldots x_n$ and hence roughly $7 \cdot M_n$ measurements are sufficient in order to amplify $P'_{\tilde{\phi}} \approx 0.81$ up to 0.96, the first method is vastly more effective.

It is now instructive (possibly enlightening as well) to compare Eq.(14) and Eq.(37) and realize that with IPEA we are really interested only in the probabilities as such. We do not try to calculate ϕ_k as a function of $P_{|+\rangle}$ and therefore we are not troubled by obstacles as those making a large portion of Eq.(9). To deal with the uncertainity in the dephasing rate γ, it is merely enough to set $\gamma \to \gamma + \Delta_\gamma$ in Eq.(37).

The problem of performing an accurate feedback for a very large n can be always easily solved by preruns of IPEA yielding at least a $(\log_2 n)$-bit estimate of $\delta g/(2g_{est})$. Therefore in the current setup (sort of atypical setup for the PEA: one qubit only + dephasing), the IPEA does not entail any fundamental barrier but a double-exponentially fast growing number of trials M_k.

This renders IPEA impractical under strong dephasing as the values of $e^{\gamma t}$ are quickly a way too large. The Naive PEA grows only single-exponentially in this respect since time t is fixed there.

C. Accumulation-enhanced Slope Adaptive PEA

The Accumulation-enhanced Slope Adaptive PEA (APEA) is a derivative of the IPEA and Naive PEA combining the crucial ideas of bit-by-bit estimation, phase accumulation and gradual adaptation towards the optimal working point.

The core idea can be outlined as follows. Let us write

$$\frac{g}{g_{est}} = \phi_0.\phi_1\phi_2\ldots\phi_\ell\phi_{\ell+1}\ldots ,$$

where $\ell = \lfloor -\log_2(\Delta_g/(2g_{est}))\rfloor$, and let the time be set to

$$t = \frac{\pi 2^k}{g_{est}} + \frac{\pi}{2g_{est}} . \tag{38}$$

Then for an integer k such that $2 \le k \le \ell - 3$ the product gt is close to a multiple of $\pi/2$,

$$gt\ (\mathrm{mod}\pi) = \frac{\pi}{2}\Big(1 \pm \frac{1}{2^{l-k-2}}\Big) , \tag{39}$$

and important bits $\phi_\ell, \phi_{\ell+1}, ..$ from the ratio g/g_{est} are accessible for naive style of measurement with $N < 2^\ell$. The Naive PEA as introduced in Section II A always requires $N \ge 2^\ell$ in order to extract non-trivial information. In accordance with the Naive PEA, we introduce a quality called the distance from the optimal working point, now denoted ς, and with respect to Eq.(39) we write $\varsigma = \ell - k - 1$ (in the Naive PEA $\varsigma = \ell$). Constraints imposed on k ensure that $\varsigma > 1$, in fact $\varsigma \ge 2$ since we work with integers, and that the shift due to $\pi/(2g_{est})$ has an effect of similar quality as if the shift was caused by an optimal value $\pi/(2g)$.

For illustration let us assume $g > g_{est}$, then the expression for the expectation value can be written as

$$\begin{aligned}\langle X\rangle &= \cos(gt)\\ &= \cos(2\pi 0.000\ldots 0\phi_\ell\phi_{\ell+1}\ldots\\ &\quad +2\pi 0.01\underbrace{0\ldots\ldots}_{\varsigma\text{bits}}\ldots\ldots 0\phi_\ell\phi_{\ell+1}\ldots) .\end{aligned} \tag{40}$$

Now, an estimate of the ratio g/g_{est} with precission $\pm 1/N$ where $N = 2^\varsigma$ yields an updated estimate of g such that $\ell \to \ell + 1$. We have gained one bit of precission in our knowledge of g.

If we are to perform a next step of APEA and obtain an additional bit we face a choice. Either to increase k by one and hereby set ς to the value it had in the previous step (non-adaptive choice), or to keep k fixed (adaptive choice). The right choice resulting in less measurements depends crucially on the strength of dephasing rate γ and its uncertainity Δ_γ.

Let us expand Eq.(14) in terms of k and ς (for simplicity time t is reduced to its vastly dominative term):

$$M \ge \Big(\frac{1}{\cos(\pi/2^\varsigma)}\frac{2e^{\gamma(\pi/g_{est})2^k}2^\varsigma}{1 - 2^\varsigma(\pi/g_{est})2^k\Delta_\gamma\tan(\pi/2^\varsigma)}\Big)^2 . \tag{41}$$

It is evident that it is advantageous to keep increasing k step by step only as long as $\exp(\gamma(\pi/g_{est})2^k) \le 2$ and $2^\varsigma(\pi/g_{est})2^k\Delta_\gamma\tan(\pi/2^\varsigma) \le 1/2$. Otherwise the nominator and/or denominator of the right factor causes too rapid growth of M. This, together with previously given constrains on k, gives us the following glorious range for k thanks to which APEA outperforms Naive PEA,

$$2 \leq k \leq \min\{ -\log_2(\frac{\pi}{g_{est}}\gamma) + \log_2(\ln 2), \\ -\log_2(\frac{\pi}{g_{est}}\Delta_\gamma) - 3,\, \ell - 3\}\,, \qquad (42)$$

if non-empty. In order to express Eq.(42) in a readable form we have used a very tight approximation $\tan(\pi/2^\varsigma) \lessapprox 2^{-\varsigma+2}$ for $\varsigma \geq 2$. Once k is saturated to its maximal value, further steps of APEA can be only adaptive.

Regarding the left factor in Eq.(41), the distance $\varsigma \geq 2$ is sufficient to keep this factor small. In fact $\varsigma = 2$ is the optimal minimal integer value since 2^ς takes part in the nominator of the right factor of M.

The last thing we need to resolve is the smallest value of ℓ for which APEA can be executed. Following the range for k we get $\ell \geq 5$. That is the initial ratio Δ_g/g_{est} have to be smaller or equal to $1/2^4$.

- A nice property of APEA is that at every step we have more and more accurate estimate of g (ℓ grows) irrespectively of whether we did adaptive or non-adaptive step. So we can perform more accurate R_x rotation.

- In fact Naive PEA and IPEA can also be performed sort of 'adaptively' aiming at only one new bit of information per 'step'. It just feels a bit unnatural since they should be think of as two very different approaches. And APEA is then their convergent derivative.

III. NUMERICAL RESULTS AND DISCUSSION

- Results: nice 3D plots

- Number of measurements vs. real time costs: since the waiting time proportional to 2^k is still small compared to the time the measurement needs → the total number of measurements does reflect real time costs! Accumulation is not a let down in this case.

- Discussion: which approach works best under which circumstances; implications for real world applications (how many bits do they have today vs. have many bits they can have - how does this change their live.

Table of brief numerical results for $g_{est} = 1, \Delta_g = 0.125, \delta g/g_{est} \leq 1/2^3, \gamma = 0.01, \delta\gamma = 0.001$:

#bits	Naive PEA	IPEA	APEA
4	1 110	30	
5	4 484	51	183
6	18 301	177	444
7	76 283	6 462	992
8	332 238	19 350 000	2 172
9	1 594 084	190000000000000	6 300
10	9 729 122		22 250
11	138 793 070		85 560
12	not possible wtht adapt.		338 300

Naive PEA: From the 10th bit the influence of Δ_γ boosts, see Eq.(14). The 12th bit and followers cannot be obtained since Eq.(13) is not satisfied.

- Notice the similarity between 8-bit NPEA and 12-bit APEA. This is because k_{max} is 4 in APEA.

- IPEA seems like a really bad option, but just try to change the dephasing rate $\gamma \to \gamma/10$ and it will shine like swedish candles :-)

IV. CONCLUSIONS

- Just an ordinary summary of what has been done

V. APPENDIX

A. Off-equatorial plane fluctuations

- The problem of R_x rotation depends on the actual qubit implementation. As I understood for some qubit it playes big role for some it does not. For this reason the core of the paper is implementation independent.

- As far as we have discussed with Göran the R_x rotation do not change the main shape of the signal. It is only shifted towards the excited state and amplitude is smaller. See the attached plot x-rotation.png. The shift can be determined as soon as qubit dies, can it? Small amplitude is bigger problem, M grows! Adaptation should be used for sure. What is the scale on the x-axis?

B. Asymmetric measurement process

- Our older notes aka what to do if your measurement device tend to prefer one answer. I don't have the latex source. Only the pdf file but it is not a big deal. The source disappeared with Göran's crashed harddrive.

C. Auxiliary plots

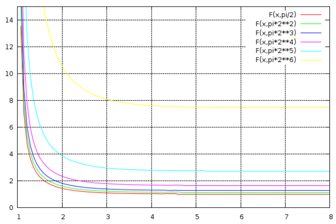

FIG. 1: $F(\ell, t) = e^{\gamma t}/\cos(\pi/2^{\ell}); \quad \gamma = 0.01.$

VI. ACKNOWLEDGEMENTS

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